Existentially valid formulas corresponding to some normal modal logics

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Existential validity

- Given a modal logic **L** with relational semantics, formula A is
  - valid iff $\forall^{\mathbf{L}\text{-frame}}\mathcal{F}\ \ \forall^{\mathcal{F}\text{-model}}\mathcal{M}\ \ \forall w \in \mathcal{M}\ \ w \Vdash A$.
  - $\exists$-valid iff $\forall^{\mathbf{L}\text{-frame}}\mathcal{F}\ \ \forall^{\mathcal{F}\text{-model}}\mathcal{M}\ \ \exists w \in \mathcal{M}\ \ w \Vdash A$.
- Examples: valid formulas, $\Box\Diamond\top$, $\Box(\Box A \to A)$.
- Humberstone [Hum08]: the set of all existentially valid formulas can be axiomatized as theorems of **K** with a new rule: $\Diamond A \vee A \ /\ A$.
- We denote the set of all formulas $\exists$-valid on all **L**-frames by $\mathbf{L}^{\exists}$.
- Our result: relationships between $\mathbf{L}^{\exists}$ for the fifteen traditional **L**? (posed by Zolin [Zol15])

Logics and frame conditions

Name	Axiom	Frame condition
T	$\Box p \rightarrow p$	xRx
B	$p \rightarrow \Box\Diamond p$	if xRy then yRx
4	$\Box p \rightarrow \Box\Box p$	if xRy and yRz then xRz
5	$\Diamond p \rightarrow \Box\Diamond p$	if xRy and xRz then yRz
D	$\Box p \rightarrow \Diamond p$	there is y s.t. xRy

- Combining these results in the fifteen well known logics.
- Which of these share their $\exists$-logics?

$\exists$-validity: inclusions

Lemma

If $\mathbf{L}_1 \subseteq \mathbf{L}_2$, then $\mathbf{L_1^\exists} \subseteq \mathbf{L_2^\exists}$.

- For example, $\mathbf{S4}^\exists \subseteq \mathbf{S5}^\exists$.
- If A is $\exists$-valid in $\mathbf{L}_1$, every $\mathbf{L}_1$-model satisfies A at some point. Every $\mathbf{L}_2$-model is an $\mathbf{L}_1$-model.
- Denote global truth of A in $\mathcal{M}$ by $\mathcal{M} \Vdash A$.

Lemma (global finite model property)

If $\mathcal{M} \Vdash A$, and $\mathcal{M}$ has some of the properties (T), (B), (4), (5) or (D), then there is a finite model $\mathcal{M}'$ having the same properties and such that $\mathcal{M}' \Vdash A$.

$\exists$-validity: inclusions

Lemma (main)

Let $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *be two of the fifteen logics obtained by extending* $\mathbf{K}$ *with some of the formulas* T, B, 4, 5 *and* D.
Assume that every finite $\mathbf{L}_1$*-frame has a generated* $\mathbf{L}_2$*-subframe.*
Then $\mathbf{L_2}^{\exists} \subseteq \mathbf{L_1}^{\exists}$.

- Proof based on [TK91], where it was proved that **S4**-consistency coincides with **S5**-consistency.
- Each **S4**-frame has a generated **S5**-subframe (generated by the set of maximal worlds).
- So, a point in this subframe that witnesses $A \in \mathbf{S5}^{\exists}$, also witnesses that $A \in \mathbf{S4}^{\exists}$.

$\exists$-validity: inclusions

Lemma

$$\mathbf{K4}^{\exists} = \mathbf{K5}^{\exists} = \mathbf{K45}^{\exists} = \mathbf{K4B}^{\exists}.$$

- We have $\mathbf{K4}^{\exists}, \mathbf{K5}^{\exists} \subseteq \mathbf{K45}^{\exists} \subseteq \mathbf{K4B}^{\exists}$. The remaining inclusions are proved by finding the right subframes and applying the previous lemma.

Lemma

$$\mathbf{D4}^{\exists} = \mathbf{D5}^{\exists} = \mathbf{S4}^{\exists} = \mathbf{S5}^{\exists} = \mathbf{D45}^{\exists}.$$

- Proved analogously (**D**-frames have only **D**-subframes), and by using the fact that $\mathbf{S4}^{\exists} = \mathbf{S5}^{\exists}$.

∃-validity: the non-inclusions

- So, we can consider only the following eight logics:

K, **KB**, **K4**, **D**, **T**, **DB**, **TB**, **D4**.

- For $F \in \{4, T, B\}$ denote $F^+ := F \wedge \Box F \wedge \Box\Box F$.
- If $\mathcal{F}$ has (F), then $\mathcal{F} \vDash F^+$.

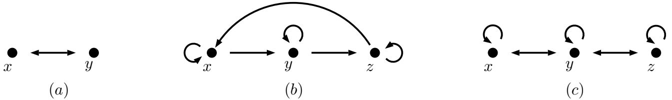

Figure: Frame (a) satisfies (B) and (D), but not (T), (4) and (5). Frame (b) satisfies (D) and (T), but not (B), (4) and (5). Frame (c) satisfies (B), (D) and (T), but not (4) and (5).

∃-validity: the non-inclusions

Lemma

We have: (i) $\mathbf{D}^{\exists} \not\subseteq \mathbf{K4}^{\exists}$, *(ii)* $\mathbf{T}^{\exists} \not\subseteq \mathbf{DB}^{\exists}$, *(iii)* $\mathbf{KB}^{\exists} \not\subseteq \mathbf{T}^{\exists}$, *and (iv)* $\mathbf{K4}^{\exists} \not\subseteq \mathbf{TB}^{\exists}$.

- To see *(i)*, take any one-point frame. It is a **K4**-frame, but does not satisfy $\Diamond\top \in \mathbf{D}^{\exists}$. So, $\Diamond\top \notin \mathbf{K4}^{\exists}$.
- To see *(ii)*, note that (a) $(x \longleftrightarrow y)$ is a **DB**-frame. Let $V(p) := \{x\}$. Now, T^{+} is not satisfiable. However, $T^{+} \in \mathbf{T}^{\exists}$.

Corollary

We have:

1. *If* $\mathbf{L_1} \in \{\mathbf{K}, \mathbf{KB}, \mathbf{K4}\}$ *and* $\mathbf{L_2} \in \{\mathbf{D}, \mathbf{T}, \mathbf{DB}, \mathbf{TB}, \mathbf{D4}\}$, *then* $\mathbf{L_2}^{\exists} \not\subseteq \mathbf{L_1}^{\exists}$.
2. *If* $\mathbf{L_1} \in \{\mathbf{K}, \mathbf{KB}, \mathbf{D}, \mathbf{DB}\}$ *and* $\mathbf{L_2} \in \{\mathbf{K4}, \mathbf{T}, \mathbf{TB}, \mathbf{D4}\}$, *then* $\mathbf{L_2}^{\exists} \not\subseteq \mathbf{L_1}^{\exists}$.
3. *If* $\mathbf{L_1} \in \{\mathbf{K}, \mathbf{D}, \mathbf{T}\}$ *and* $\mathbf{L_2} \in \{\mathbf{KB}, \mathbf{K4}, \mathbf{DB}, \mathbf{TB}, \mathbf{D4}\}$, *then* $\mathbf{L_2}^{\exists} \not\subseteq \mathbf{L_1}^{\exists}$.
4. *If* $\mathbf{L_1} \in \{\mathbf{K}, \mathbf{KB}, \mathbf{D}, \mathbf{T}, \mathbf{DB}, \mathbf{TB}\}$ *and* $\mathbf{L_2} \in \{\mathbf{K4}, \mathbf{D4}\}$, *then* $\mathbf{L_2}^{\exists} \not\subseteq \mathbf{L_1}^{\exists}$.

∃-validity

$$\mathbf{K4}^{\exists} = \mathbf{K5}^{\exists} = \mathbf{K45}^{\exists} = \mathbf{K4B}^{\exists}.$$

$$\mathbf{D4}^{\exists} = \mathbf{D5}^{\exists} = \mathbf{S4}^{\exists} = \mathbf{S5}^{\exists} = \mathbf{D45}^{\exists}.$$

$$\begin{array}{ccccc}
\mathbf{T}^{\exists} & \rightarrow & \mathbf{TB}^{\exists} & \rightarrow & \mathbf{D4}^{\exists} \\
\uparrow & & \uparrow & & \\
\mathbf{D}^{\exists} & \rightarrow & \mathbf{DB}^{\exists} & & \Bigg\uparrow \\
\uparrow & & \uparrow & & \\
\mathbf{K}^{\exists} & \rightarrow & \mathbf{KB}^{\exists} & \rightarrow & \mathbf{K4}^{\exists}
\end{array}$$

Figure: Arrows represent proper subsets.

Papers mentioned

Lloyd Humberstone, *Modal formulas true at some point in every model*, Australasian Journal of Logic **6** (2008), 70–82.

Michael Tiomkin and Michael Kaminski, *Nonmonotonic default modal logics*, Journal of the ACM **38** (1991), 963–984.

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