Decidability of some interpretability logics

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Interpretability logic

- Intepretability logics have a binary modal operator $\rhd$.
- Basic interpretability logic **IL**:

	classically valid formulas (in the new language, $\Box, \Diamond, \rhd$);
K	$\Box(A \to B) \to (\Box A \to \Box B)$;
Löb	$\Box(\Box A \to A) \to \Box A$;
J1	$\Box(A \to B) \to A \rhd B$;
J2	$(A \rhd B) \ \wedge \ (B \rhd C) \to A \rhd C$;
J3	$(A \rhd C) \ \wedge \ (B \rhd C) \to A \vee B \rhd C$;
J4	$A \rhd B \to (\Diamond A \to \Diamond B)$;
J5	$\Diamond A \rhd A$.

  - rules: modus ponens and necessitation $A/\Box A$.

(parentheses priority: $\neg, \Box, \Diamond;\ \wedge, \vee;\ \rhd;\ \to, \leftrightarrow$)

Models

- Semantics: extend the usual relational (Kripke) model.
- Veltman model: $\mathcal{M} = \langle W, R, \underline{\{S_w \ : \ w \in W\}}, V\rangle$, where:
  1. $W \neq \emptyset$;
  2. R^{-1} is well-founded (no $x_0 R x_1 R x_2 R \dots$ chains);
  3. R is transitive;
  4. $S_w \subseteq R(w)^2$ is reflexive, transitive, contains $R \cap R(w)^2$ ($wRuRv$ implies $uS_w v$);
  5. $V : Prop \to \mathcal{P}(W)$.
- Truth of a formula $F \rhd G$ ("F interprets G") in a world $w \in \mathcal{M}$:

$$w \Vdash F \rhd G \quad :\Leftrightarrow \quad \forall x \in R(w) : \quad x \Vdash F \Rightarrow \exists y \in S_w(x) : \ y \Vdash G.$$

- **IL**-frame (Veltman frame) is a triple $\mathcal{F} = \langle W, R, \{S_w \ : \ w \in W\}\rangle$.
- We have:

$$\mathbf{IL} \vdash F \quad \Leftrightarrow \quad \forall \mathcal{F} : \ \mathcal{F} \models F.$$

Frame conditions

- Some extensions of **IL**:

$\mathbf{IL}M_0$	$\mathbf{IL} + A \rhd B \to \Diamond A \wedge \Box C \rhd B \wedge \Box C$
$\mathbf{IL}W$	$\mathbf{IL} + A \rhd B \to A \rhd B \wedge \Box\neg A$
$\mathbf{IL}W^*$	$\mathbf{IL} + A \rhd B \to B \wedge \Box C \rhd B \wedge \Box C \wedge \Box\neg A$

- $\mathbf{IL}W^* = \mathbf{IL}M_0 W \subseteq \mathbf{IL}(All)$
- These logics are complete w.r.t. certain classes of frames:

(M_0)	$wRuRxS_w v \Rightarrow R(v) \subseteq R(u)$;
(W)	$S_w \circ R$ is reverse well-founded for each w;
(W^*)	(M_0) and (W).

- **IL**W-frame is **IL**-frame that satisfies (W) etc.

Proving decidability

- Let's focus on **IL**.
- FMP: if $x \nVdash F$, then there is finite $\mathcal{M}$ and $x' \in \mathcal{M}$ s.t. $x' \nVdash F$.
- Decision procedure: simultaneously do two things:
  - Enumerate the (countable) set of all **IL**-proofs.
  - Enumerate the (countable) set of (descriptions of) finite **IL**-models.
- The usual way of proving FMP is by filtrations.

Filtrations on **IL**-models

- Let Γ contain A, closed under subformulas.
- Assume $\sim$ is an equivalence relation on W, $\sim \subseteq \equiv_\Gamma$.
- For any $V \subseteq W$, define $\widetilde{V} = \{[v] \mid v \in V\}$.
- We define the rest of $\widetilde{M}$ as follows.
- $\widetilde{R} = \{([w],[u]) \mid wRu, \exists \Box C \in \Gamma : w \nVdash \Box C, u \Vdash \Box C\}$.
- $[u]\widetilde{S}_{[w]}[v]$ if and only if $[u],[v] \in \widetilde{R}([w])$, and for <u>all/some</u> $w' \in [w]$ and <u>some</u> $u' \in [u]$ such that $w'Ru'$ we have $u'S_{w'}v'$ for some $v' \sim v$.
- Define $\Vdash$ so that x and $[x]$ agree on variables in Γ.
- We'll write R, S instead of $\widetilde{R}$, $\widetilde{S}$ when context allows.
- Problem: we lose transitivity of $S_{[w]}$.
  $w \to \{u \rightsquigarrow v_1 \sim v_2 \rightsquigarrow z\}$, $[w] \to \{[u] \rightsquigarrow [v] \rightsquigarrow [z]\}$

Filtrations on **IL**-models (2)

- Let Γ contain A, closed under subformulas (and some more technical conditions).
- Assume $\sim$ is an equivalence relation on W, $\sim \subseteq \equiv_\Gamma$.
- For any $V \subseteq W$, define $\widetilde{V} = \{[v] \mid v \in V\}$.
- We define the rest of $\widetilde{M}$ as follows.
- $\widetilde{R} = \{([w],[u]) \mid wRu, \exists \Box C \in \Gamma : w \nVdash \Box C, u \Vdash \Box C\}$.
- $[u]\widetilde{S}_{[w]}[v]$ if and only if $[u],[v] \in \widetilde{R}([w])$, and for some/all $w' \in [w]$ and all $u' \in [u]$ such that $w'Ru'$ we have $u'S_{w'}v'$ for some $v' \sim v$.
- Define $\Vdash$ so that x and $[x]$ agree on variables in Γ.
- We'll write R, S instead of $\widetilde{R}$, $\widetilde{S}$ when context allows.
- Problem: we lose S_w-successors that don't agree enough.
  $w \to \{v_1[X] \leftsquigarrow u_1 \sim u_2 \rightsquigarrow v_2[\neg X]\}$,
  $[w] \to \{[u] \rightsquigarrow \quad ? \quad \}$

Generalized models

- In the last example, ideally $[u] \rightsquigarrow \{v_1, v_2\}$.
- Generalized **IL**-models (generalized Veltman models).
- $\mathcal{M} = \langle W, R, \underline{\{S_w \ : \ w \in W\}}, V \rangle$, where:
  1. $W \neq \emptyset$;
  2. R^{-1} is well-founded (no $x_0 R x_1 R x_2 R \ldots$ chains);
  3. R is transitive;
  4. $S_w \subseteq R(w) \times (2^{R(w)} \setminus \{\emptyset\})$ is:
     - quasi-reflexive $\quad uS_w\{u\}$;
     - quasi-transitive $\quad uS_w\{v_i \mid i \in I\}$ and $v_i S_w Z_i \Rightarrow uS_w \bigcup\{Z_i \mid i \in I\}$;
     - contains $R \cap R(w)^2 \quad wRuRv$ implies $uS_w\{v\}$;
     - is monotonous $\quad uS_w V \Rightarrow uS_w V', V \subseteq V'$
  5. $V : Prop \to \mathcal{P}(W)$.
- Truth of a formula $F \rhd G$ ("F interprets G") in a world $x \in \mathcal{M}$:

$$w \Vdash F \rhd G \quad :\Leftrightarrow \quad \forall x \in R(w) : \ \ x \Vdash F \Rightarrow \exists V \in S_w(x) : \ V \Vdash G.$$

- $V \Vdash G$ stands for $v \Vdash G$ for all $v \in V$.

Filtration property

- $\widetilde{M} = \langle \widetilde{W}, \widetilde{R}, \widetilde{S}_{[w]}, \Vdash \rangle$.
- $\widetilde{W} = \{[w] \mid w \in W\}$.
- $\widetilde{R} = \{([w],[u]) \mid wRu, \exists \Box C \in \Gamma : w \nVdash \Box C, u \Vdash \Box C\}$.
- $[u]\widetilde{S}_{[w]}\widetilde{V}$ if and only if $\{[u]\}, \widetilde{V} \subseteq R([w])$, and for all $w' \in [w]$ and all $u' \in [u]$ such that $w'Ru'$ we have $u'S_{w'}V(w',u')$ for some $V\widetilde{(w',u')} \subseteq \widetilde{V}$.
- Forcing relation compatible with $\mathcal{M}$.
- $w \to \{\{v_1[X]\} \leftsquigarrow u_1 \sim u_2 \rightsquigarrow \{v_2[\neg X]\}\}$,
  $[w] \to \{[u] \rightsquigarrow \{[v_1],[v_2]\}\}$
- Assume $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a generalized model (depends on $\sim$).
- Do we have $w \Vdash F \iff [w] \Vdash F$?

- Denote $[A]_w = \{x \in R[w] \mid x \Vdash A\}$.

Lemma

Let $w \nVdash A \rhd B$. There is a maximal $u \in [A]_w$ such that

$$uS_w V \Rightarrow V \nVdash B.$$

We also have $u \nVdash \Diamond A, B$.

Proof.

Existence: definition of $\Vdash$. Maximality: R is conversely well-founded. Since $uS_w\{u\}$, obviously $u \nVdash B$. Suppose $u \Vdash \Diamond A$. Then $uRv \Vdash A$. Since $uS_w\{v\}$, by quasi-transitivity we have $S_w(v) \subseteq S_w(u)$. Contradiction with maximality of u. $\square$

Theorem

$w \Vdash F \iff [w] \Vdash F.$

Proof.

Induction on F.

$\Leftarrow$ Assume $w \nVdash A \rhd B$. Lemma: there is a maximal $u \in [A]_w$ such that $uS_wV \Rightarrow V \nVdash B$; and $u \nVdash \Diamond A$.
We have $w \Vdash \Diamond A$, and since $u \nVdash \Diamond A$, $[w]R[u]$.
Let $\widetilde{V}$ arbitrary s.t. $[u]S_{[w]}\widetilde{V}$. In particular, uS_wV' for some $\widetilde{V'} \subseteq \widetilde{V}$. Since $V' \nVdash B$, by IH, $\widetilde{V'} \nVdash B$. Therefore $\widetilde{V} \nVdash B$.

$\square$

Theorem

$w \Vdash F \iff [w] \Vdash F.$

Proof.

Induction on F.

$\Rightarrow$ Assume $w \Vdash A \rhd B$. Assume $[w]R[u] \Vdash A$. We construct $\widetilde{V}$ s.t. $[w]R[u]S_{[w]}\widetilde{V} \Vdash B$.
Let $w' \in [w], u' \in [u], wRu$. Since $w' \sim w$, $w' \Vdash A \rhd B$, therefore for some $V(w', u')$, $u' S_{w'} V(w', u') \Vdash B$.
For each point $v \in V(w', u')$, put $Z_v = \{v\}$ if $v \nVdash \Diamond B$.
Otherwise, $Z_v = \{m\}$, where m is arbitrary maximal world from $[B]_v$. Now, $u' S_w Z_v$, so by quasi-transitivity, $vS_w \bigcup_v Z_v \Vdash \Box\neg B$.
Put $V := \bigcup_{w' \in [w], u' \in [u], w' R u', v \in V(w', u')} Z_v$. By IH, $\widetilde{V} \Vdash B, \Box\neg B$.
It remains to show that $\widetilde{V} \subseteq R([w])$. This requires $\exists C : \Box C \in \Gamma, [w] \nVdash \Box C, \widetilde{V} \Vdash \Box C$. Take $C = \neg B$.

$\square$

- So, if $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a model at all, then it is a filtration of $\mathcal{M} = \langle W, R, S, \Vdash \rangle$.
- Is it a model (does it satisfy quasi-transitivity etc.)? Depends on what $\sim$ is.
- Ideally, x and $[x]$ are structurally similar, so that quasi-transitivity etc. is preserved.
- So, each $y \sim x$ should be structurally similar to x.

Definition

A *bisimulation* between generalized $\mathbf{IL}$-models $\langle W, R, \{S_w : w \in W\}, \Vdash\rangle$ and $\langle W', R', \{S'_{w'} : w' \in W'\}, \Vdash\rangle$ is any $Z \subseteq W \times W'$, $Z \neq \emptyset$:

(at) if wZw' then $w \Vdash p \iff w' \Vdash p$;

(forth) if wZw' and wRu, then there exists $u' \in R'(w')$ with uZu' and for all $V' \in S'_{w'}(u')$ there is $V \in S_w(u)$ such that for all $v \in V$ there is $v' \in V'$ with vZv';

(back) if wZw' and $w'R'u'$, then there exists $u \in R(w)$ such that uZu' and for all $V \in S_w(u)$ there is $V' \in S'_{w'}(u')$ such that for all $v' \in V'$ there is $v \in V$ with vZv'.

- By induction on F, if x and y are bisimilar (w.r.t. any bisimulation), $x \Vdash F \iff y \Vdash F$.
- Union of bisimulations (over generalized models) is itself a bisimulation (*Vrgoč and Vuković, 2010*).
- In particular, there is a largest (auto)bisimulation $Z \subseteq W^2$.

- Denote by $\sim$ the largest bisimulation on W^2.

  (equivalently, denote $x \sim y$ if there is any bisimulation at all which equates x and y)

Theorem

$\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ *is a model.*

Proof.

We should check: (1) $\widetilde{W} \neq \emptyset$, (2) $\widetilde{R}^{-1}$ is well-founded, (3) $\widetilde{R}$ is transitive, (4) $\widetilde{S}_{[w]} \subseteq \widetilde{R}([w]) \times (2^{\widetilde{R}([w])} \setminus \{\emptyset\})$ (5) is quasi-reflexive $[u]\widetilde{S}_{[w]}\{[u]\}$, (6) quasi-transitive $[u]\widetilde{S}_{[w]}\{[v_i] \mid i \in I\}$ and $[v_i]\widetilde{S}_{[w]}Z_i \Rightarrow [u]\widetilde{S}_{[w]} \bigcup\{Z_i \mid i \in I\}$, (7) contains $\widetilde{R} \cap \widetilde{R}([w])^2$ $[w]\widetilde{R}[u]\widetilde{R}[v]$ implies $[u]\widetilde{S}_{[w]}\{[v]\}$, (8) is monotonous $[u]\widetilde{S}_{[w]} V \Rightarrow [u]\widetilde{S}_{[w]} V', V \subseteq V'$ $\square$

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(3). Assume $[w]R[u]R[v]$. Then (w.l.o.g.) $wRu \sim u'Rv$. Now (back) implies there is $v' \in R(u)$, $v' \sim v$. So $wRuRv'$, thus wRv'. Since $[w]R[u]$, there is A s.t. $w \nVdash \Box\neg A$, $u \Vdash \Box\neg A$. So, also $v' \Vdash \Box\neg A$. But then $[w]R[v']$. Since $v' \sim v$, $[w]R[v]$.

Proof.

We should check: (1) $\widetilde{W} \neq \emptyset$, (2) $\widetilde{R}^{-1}$ is well-founded, (3) $\widetilde{R}$ is transitive, (4) $\widetilde{S}_{[w]} \subseteq \widetilde{R}([w]) \times 2^{\widetilde{R}([w])}$ (5) is quasi-reflexive $[u]\widetilde{S}_{[w]}\{[u]\}$, (6) quasi-transitive $[u]\widetilde{S}_{[w]}\{[v_i] \mid i \in I\}$ and $[v_i]\widetilde{S}_{[w]}Z_i \Rightarrow [u]\widetilde{S}_{[w]} \bigcup\{Z_i \mid i \in I\}$, (7) contains $\widetilde{R} \cap \widetilde{R}([w])^2$ $[w]\widetilde{R}[u]\widetilde{R}[v]$ implies $[u]\widetilde{S}_{[w]}\{[v]\}$, (8) is monotonous $[u]\widetilde{S}_{[w]}V \Rightarrow [u]\widetilde{S}_{[w]}V', V \subseteq V'$.

(3). Assume $[w]R[u]R[v]$. Then (w.l.o.g.) $wRu \sim u'Rv$. Now (back) implies there is $v' \in R(u)$, $v' \sim v$. So $wRuRv'$, thus wRv'. Since $[w]R[u]$, there is A s.t. $w \nVdash \Box\neg A$, $u \Vdash \Box\neg A$. So, also $v' \Vdash \Box\neg A$. But then $[w]R[v']$. Since $v' \sim v$, $[w]R[v]$.

(7) Assume $[w]\widetilde{R}[u]\widetilde{R}[v]$. We already know $[w]\widetilde{R}[u]$ and $[w]\widetilde{R}[v]$. Let $w' \sim w, u' \sim u$ such that $w'Ru'$. Since $u' \sim u$, (back) implies there is $v' \sim v$ such that $w'Ru'Rv'$. So for arbitrary $w' \sim w, u' \sim u$ there is v' s.t. $u'S_{w'}\{v'\}$ and indeed $[v'] \in \{[v]\}$. $\square$

- Thus, if $\sim$ is the largest bisimulation on W^2, then $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a model, and a filtration.

  We were trying to prove finite model property; is this a finite model?
- Each $\widetilde{R}$-transition eliminates at least one $\diamond$-formula from Γ; so height is finite.
- Still, branching factor might be infinite.

Definition

A *n-bisimulation* between **IL**-models $\langle W, R, \{S_w : w \in W\}, \Vdash\rangle$ and $\langle W', R', \{S'_{w'} : w' \in W'\}, \Vdash\rangle$ is any sequence $Z_n \subseteq \cdots \subseteq Z_0 \subseteq W \times W'$:

(at) if wZ_0w' then $w \Vdash p \iff w' \Vdash p$;

(forth) if wZ_nw' and wRu, then there exists $u' \in R'(w')$ with $uZ_{n-1}u'$ and for all $V' \in S'_{w'}(u')$ there is $V \in S_w(u)$ such that for all $v \in V$ there is $v' \in V'$ with $vZ_{n-1}v'$;

(back) if wZ_nw' and $w'R'u'$, then there exists $u \in R(w)$ such that $uZ_{n-1}u'$ and for all $V \in S_w(u)$ there is $V' \in S'_{w'}(u')$ such that for all $v' \in V'$ there is $v \in V$ with $vZ_{n-1}v'$.

- Since height of $\mathcal{M}$ is bounded by $|\Gamma|$, worlds are $|\Gamma|$-bisimilar iff bisimilar.

- Put $u \equiv_n v$ if u and v agree on all formulas with at most n nested modalities.
- From now on, assume $Prop := Prop \cap \Gamma$.
- Now there are only finitely many formulas of modal depth up to $|\Gamma|$ (finitely many up to local equivalence).
- Denote $Th_n\ w$ the set of all formulas F with modal depth up to $|\Gamma|$ and $w \Vdash F$.

Lemma

$u \sim_n v \iff u \equiv_n v.$

Proof.

- $\Rightarrow$ Induction on F.
- $\Leftarrow$ Induction on n. Step: assume (forth) doesn't hold. Then there is $u \in R(w)$:

$$(\forall u' \sim_{n-1} u, u' \in R(w'))(\exists V'(u') \in S_{w'}(u'))(\forall V \in S_w(u))$$
$$(\exists v(u', V) \in V)(\forall v' \in V'(u'))v(u', V) \nsim_{n-1} v'.$$

Put $B_V := \bigwedge_{u' \sim_{n-1} u, u' \in R(w')} Th_{n-1}\, v(u', V)$. Put $B := \bigwedge_{V \in S_w(u)} \neg B_V$. For all $u' \sim u$, we have $V'(u') \Vdash B$ (because $v(u', V) \nsim_{n-1} v'$).
Let $A := Th_{n-1}\, u$. Now $w' \Vdash A \rhd B$. Since $w \equiv_n w'$, then $w \Vdash A \rhd B$. Contradiction.

$\square$

- Denote $\mathcal{N} = \widetilde{\mathcal{M}}$.
- For $x, y \in \mathcal{N}$, we now have $x \sim y \iff x \sim_{|\Gamma|} y \iff x \equiv_{|\Gamma|} y$.
- There are obviously only finitely many worlds in $\mathcal{M}/\equiv_{|\Gamma|}$.
- Since $\equiv_{|\Gamma|} = \sim_{|\Gamma|}$, $\widetilde{\mathcal{N}}$ (that is, $\widetilde{\widetilde{\mathcal{M}}}$) has only finitely many worlds.
- Thus we have FMP for **IL**.

Extending to **IL**X

- To prove FMP, given **IL**X that is complete w.r.t. class of Veltman frames that satisfy property $\mathcal{C}$, we need to fill in the following:
  1. What is the (generalized) frame condition $\mathcal{G}$ of X?
  2. Is **IL**X complete w.r.t. to the class of $\mathcal{G}$-frames?
  3. Does $\widetilde{\mathcal{M}}$ have $\mathcal{G}$ if $\mathcal{M}$ has $\mathcal{G}$?
- For popular choices of X (except for W, W*), 1 is known; and 2 usually reduces to completeness w.r.t. $\mathcal{C}$ (for each VM take the natural GVM, i.e. $uS_w v \Rightarrow uS_w\{v\}$).

Logic $\mathbf{IL}\mathsf{M}_0$

- $\mathbf{IL}\mathsf{M}_0$ is $\mathbf{IL} + A \rhd B \to \Diamond A \wedge \Box C \rhd B \wedge \Box C$.
- Frame condition (M_0):

$$wRuRxS_w vRz \Rightarrow uRz.$$

- Frame condition $(M_0)_{gen}$:

$$wRuRxS_w V \Rightarrow (\exists V' \subseteq V)(uS_W V' \ \& \ R(V') \subseteq R(u)).$$

- For each VM with (M_0), there is a natural GVM (for $xS_w y$, $xS_W\{y\}$) with $(M_0)_{gen}$.
- Remains to prove $\widetilde{\mathcal{M}}$ preserves $(M_0)_{gen}$.

Theorem

If $\mathcal{M}$ has property $(M_0)_{gen}$, then $\widetilde{\mathcal{M}}$ has property $(M_0)_{gen}$.

Proof.

Let $[w]R[u]R[x]S_{[w]}\widetilde{V}$. Fix $w' \in [w], u' \in [u]$. By bisimilarity, there is $x' \sim x$, $w'Ru'Rx'$.
Since $[x]S_{[w]}\widetilde{V}$, there is $V(w', u')$ such that $x'S_{w'}V(w', u')$ and $\widetilde{V(w', u')} \subseteq \widetilde{V}$. By (M_0), there is $V'(w', u') \subseteq V(w', u')$ such that $R(V'(w', u')) \subseteq R(u')$.
Choose such $V'(w', u')$ for $w' \in [w], u' \in [u]$; $V' = \bigcup V'(w', u')$.
To show $[u]S_{[w]}\widetilde{V'}$, it remains to show $R(V') \subseteq R([u])$. Take $[v] \in V'$ and any $[z] \in R([v])$, w.l.o.g. we have vRz. By definition, $v \sim v' \in V'(w', u')$ for some $v', w' \sim w, u' \sim u$. Since $v \sim v'$, $v'Rz'$ for some $z' \sim z$. We had $R(V'(w', u')) \subseteq R(u')$.
So, $z' \in R(u')$. To show $[z] \in R([u])$, there should be a formula C, $[u] \Vdash \Diamond C$, $[z] \nVdash \Diamond C$. Take such C from $[v]R[z]$.
Since $v \sim v'$, $v' \Vdash \Diamond C$ and $R(V'(w', u')) \subseteq R(u')$, we have $u' \Vdash \Diamond C$. $\square$

Logic **IL**W

- **IL**W is **IL** $+ A \rhd B \to A \rhd B \wedge \Box\neg A$.
- Frame condition (W):

$$S_w \circ R \text{ is reverse well-founded for each } w$$

- Frame condition $(W)_{gen}$?

$$(\forall w \in W)(\forall X \subseteq R(w))(\forall Z \subseteq S_w^{-1}(X), Z \neq \emptyset)(\forall z \in Z)$$
$$(\exists V \subseteq X)\Big(zS_wV \;\&\; (\forall v \in V)(R(v) \cap Z = \emptyset)\Big).$$

- $(\forall Z \subseteq S_w^{-1}(X)$ is: for all Z such that for all $z \in Z$, zS_wX)
- (Interestingly, equivalent after replacing $(\forall z \in Z)$ with $(\exists z \in Z)$; occasionally useful.)

Logic $\mathbf{IL}\mathsf{W}^*$

- $\mathbf{IL}\mathsf{W}^*$ is $\mathbf{IL} + A \rhd B \to B \wedge \Box C \rhd B \wedge \Box C \wedge \Box\neg A$.
- $\mathbf{IL}\mathsf{W}^* = \mathbf{IL}\mathsf{WM}_0$.
- Frame condition $(W^*)_{gen}$?
- Each $\mathbf{IL}\mathsf{W}^*$-frame is $\mathbf{IL}\mathsf{W}$-frame ($\mathbf{IL}\mathsf{WM}_0 \supseteq \mathbf{IL}\mathsf{W}$) and $\mathbf{IL}\mathsf{M}_0$-frame ($\mathbf{IL}\mathsf{WM}_0 \supseteq \mathbf{IL}\mathsf{M}_0$).
- Conversely, if $\mathcal{F}$ is both an $\mathbf{IL}\mathsf{W}$-frame and an $\mathbf{IL}\mathsf{M}_0$-frame, then it is an $\mathbf{IL}\mathsf{WM}_0$-frame (induction on proof length).
- So, the frame condition is:
$$(W)_{gen} \text{ and } (M_0)_{gen}.$$
- If $\mathbf{IL}W^* \nvdash F$, there is a $\mathbf{IL}\mathsf{M}_0$-, $\mathbf{IL}\mathsf{W}$-VM $\mathcal{M}$, $w \in \mathcal{M}$, s.t. $w \nVdash F$. Then there is a natural GVM $\mathcal{N}$ with similar properties. Then $\widetilde{\mathcal{N}}$ is an $\mathbf{IL}\mathsf{M}_0$-, $\mathbf{IL}\mathsf{W}$-GVM, and so an $\mathbf{IL}\mathsf{W}^*$-GVM.

Complexity

- Given X, what is comp. complexity of $\{F \mid \mathbf{IL}X \vdash F\}$?
- Since $\mathbf{GL} \subseteq \mathbf{IL}$, at least PSPACE for any natural choice of X.
- The only (?) known result: $\mathbf{IL}_0$ is PSPACE-hard.
- Our goals:
  - **IL** is in PSPACE;
  - **IL**W is in PSPACE.
- (corollary: both are PSPACE-complete)

Complexity (2)

- Let F be any non-theorem of **IL**X. By completeness, there is $\mathcal{M}$, $w \in \mathcal{M}$ s.t. $w \nVdash F$.
  1. Show that $\mathcal{M}$ can be transformed to a certain model $\mathcal{M}^f$ with some desirable properties:
     - accessibility relation (R) is a tree;
     - polynomial height;
     - polynomial branching factor;
     - S-relations should be "factorized".
  2. Show that there is an algorithm that verifies the existence of all models with such properties. For **IL**W, do additional (polynomially large) bookkeeping to ensure there are no $R \circ S_w$-loops.

Papers

- T. Perkov, M. Vuković. Filtrations of generalized Veltman models. *Mathematical Logic Quarterly*, **62**, 412–419, 2016.
- L. Mikec, T. Perkov, M. Vuković. Decidability of interpretability logics $\mathbf{IL}\mathtt{M}_0$ and $\mathbf{IL}\mathtt{W}^*$. *Submitted*.
- (complexity paper - work in progress)